Superfield Extended BRST Quantization in General Coordinates

B. GEYER[a)]*, D.M. GITMAN[b)]†, P.M. LAVROV[a),c)]‡, and P.YU. MOSHIN[b),c)]§

[a)] *Center of Theoretical Studies, Leipzig University, Augustusplatz 10/11, D-04109 Leipzig, Germany*
[b)] *Instituto de Física, Universidade de São Paulo, Caixa Postal 66318-CEP, 05315-970 São Paulo, S.P., Brazil*
[c)] *Tomsk State Pedagogical University, 634041 Tomsk, Russia*

Abstract

We propose a superfield formalism of Lagrangian BRST–antiBRST quantization of arbitrary gauge theories in general coordinates with the base manifold of fields and antifields described in terms of both bosonic and fermionic variables.

1 Introduction

The principle of extended BRST symmetry provides the basis of several Lagrangian quantization schemes for general gauge theories, including the well-known $Sp(2)$-covariant approach [1] and its different modifications, e.g., the superfield formalism [2] and the two versions of triplectic quantization [3, 4]. In order to reveal the geometric content of extended BRST symmetry, it is important to study these quantization methods in general coordinates (see, e.g., [5, 6] and references therein).

In the recent paper [6], it was shown that the geometry of the $Sp(2)$-covariant and triplectic schemes is the geometry of an even symplectic supermanifold equipped with a scalar density function and a flat symmetric connection (covariant derivative), while the geometry of the modified triplectic quantization also includes a symmetric structure (analogous to a metric tensor). The study of [6] generalizes the concept of triplectic supermanifolds, introduced in [5], to the case of base manifolds [5, 6] containing not only bosonic but also fermionic variables.

In this paper, we propose a superfield version of the quantization scheme developed in [5, 6]. The superfield description naturally involves an extension of supermanifolds used in [5, 6]. Namely, the triplectic supermanifold is extended to the complete supermanifold of variables used in the original $Sp(2)$-covariant approach. Note that in Darboux coordinates a similar extension takes place in the superfield formulation [2] of the $Sp(2)$-covariant scheme.

The paper is organized as follows. In Section 2, we propose a superfield extension of triplectic supermanifolds and introduce an operation of covariant differentiation on such supermanifolds, following the approach of our previous works [5, 6]. In Section 3, we propose a manifest realization of the (modified) triplectic algebra [3, 4] and outline a suitable quantization procedure along the lines of [5, 6]. In Section 4, we summarize the results and make concluding remarks.

We use DeWitt's condensed notation [7] and apply tensor analysis on supermanifolds [8]. Left-hand derivatives with respect to some variables x^i are denoted as $\partial_i A = \partial A/\partial x^i$. Right-hand derivatives with respect to x^i are labelled by the subscript "r", and the notation $A_{,i} = \partial_r A/\partial x^i$ is used. The covariant derivative ∇ (and other operators acting on tensor fields) is assumed to act from the right: $A\nabla$; if necessary, the action of an operator from the right is indicated by an arrow, e.g., $\overleftarrow{\nabla}$. Raising the $Sp(2)$-group indices is performed with the help of the antisymmetric second rank tensor ε^{ab} $(a, b = 1, 2)$: $\theta^a = \varepsilon^{ab}\theta_b$, $\varepsilon^{ac}\varepsilon_{cb} = \delta^a_b$. The Grassmann parity of a quantity A is denoted by $\epsilon(A)$.

*E-mail: geyer@itp.uni-leipzig.de
†E-mail: gitman@dfn.if.usp.br
‡E-mail: lavrov@tspu.edu.ru
§E-mail: moshin@dfn.if.usp.br

2 Superfield Extension of Triplectic Supermanifolds

The supervariables used in various realizations of extended BRST symmetry can be naturally combined into a set (x^i, θ^i_a, y^i), $i = 1, 2, \ldots, N = 2n$. Thus, the supermanifolds of the triplectic [3] and modified triplectic [4] quantization schemes consist of the variables $x^i = (\phi^A, \bar{\phi}_A)$ and $\theta^i_a = (\pi^A_a, \phi^*_{Aa})$, where ϕ^A are the fields of the configuration space of a general gauge theory; the antifields $\bar{\phi}_A$ are the sources of the combined BRST–antiBRST symmetry; the antifields ϕ^*_{Aa} are the sources of the BRST and antiBRST transformations; while π^{Aa} are auxiliary (gauge-fixing) fields. A superfield description [2] of extended BRST symmetry requires an extension of triplectic supermanifolds [3, 4] by the additional (external) variables $y^i = (\lambda^A, J_A)$ arising in the original $Sp(2)$-covariant scheme [1], where λ^A are auxiliary (gauge-fixing) fields, and J_A are the sources to the fields ϕ^A. The realization of extended BRST symmetry in general coordinates [6] is based on a tensor analysis on supermanifolds with coordinates (x^i, θ^i_a). In this section, we propose a superfield formulation of the analysis [6].

2.1 Superfields, Component Transformations

Let us consider a superspace spanned by space-time coordinates and an $Sp(2)$-doublet of anticommuting coordinates η^a. Any function $f(\eta)$ has a component representation,

$$f(\eta) = f_0 + \eta^a f_a + \eta^2 f_3, \quad \eta^2 \equiv \frac{1}{2}\eta_a \eta^a,$$

and an integral representation,

$$f(\eta) = \int d^2\eta' \, \delta(\eta' - \eta) f(\eta'), \quad \delta(\eta' - \eta) = (\eta' - \eta)^2,$$

where integration over η^a is given by

$$\int d^2\eta = 0, \quad \int d^2\eta \; \eta^a = 0, \quad \int d^2\eta \; \eta^a\eta^b = \varepsilon^{ab}.$$

In particular, for any superfield $f(\eta)$ we have

$$\int d^2\eta \, \frac{\partial f(\eta)}{\partial \eta^a} = 0,$$

which implies the property of integration by parts

$$\int d^2\eta \, \frac{\partial f(\eta)}{\partial \eta^a} g(\eta) = -\int d^2\eta \, (-1)^{\varepsilon(f)} f(\eta) \frac{\partial g(\eta)}{\partial \eta^a} \, ,$$

where derivatives with respect to η^a are taken from the left.

Let us now introduce a set of superfields $z^i(\eta)$, $\epsilon(z^i) = \epsilon_i$, $i = 1, ..., N$, with the component notation

$$z^i(\eta) = x^i + \eta^a \theta^i_a + \eta^2 y^i,$$

and the following distribution of Grassmann parity:

$$\epsilon(x^i) = \epsilon(y^i) = \epsilon_i, \quad \epsilon(\theta^i_a) = \epsilon_i + 1.$$

We shall identify the components (x^i, θ^i_a, y^i) with local coordinates of a supermanifold $\mathcal{N}$, $\dim \mathcal{N} = 4N$, where the submanifold $\mathcal{M}$, $\dim \mathcal{M} = 3N$, with coordinates (x^i, θ^i_a) is chosen as a *triplectic supermanifold* [5, 6]. We accordingly define the following transformations of the local coordinates:

$$\bar{x}^i = \bar{x}^i(x), \quad \bar{\theta}^i_a = \theta^j_a \frac{\partial \bar{x}^i}{\partial x^j} \, , \quad \bar{y}^i = y^i \, , \tag{1}$$

where $\bar{x}^i = \bar{x}^i(x)$ are transformations on the submanifold M, $\dim M = N$, with coordinates (x^i), called the *base supermanifold* [6]. The transformations of the coordinates (x^i, θ^i_a) are identical with

the transformations which define a triplectic supermanifold [5, 6]. The superfield derivative $\frac{\overleftarrow{\partial}}{\partial z^i(\eta)}$ with respect to variations $\delta z^i(\eta) = \delta x^i + \eta^a \delta\theta^i_a$ induced by the component transformations (1),

$$\frac{\overleftarrow{\partial}}{\partial z^i(\eta)} = \frac{\overleftarrow{\partial}}{\partial \theta^i_a}\eta_a + \frac{\overleftarrow{\partial}}{\partial x^i}\eta^2\,, \tag{2}$$

is trivial on the external variables y^i. Using the derivative (2) and the transformations (1), we can introduce a superfield extension of covariant differentiation [5, 6] on triplectic supermanifolds $\mathcal{M}$.

2.2 Superfield Extension of Triplectic Covariant Derivative

As a preliminary step, we shall discuss some elements of tensor analysis on the base supermanifold M, referring for a detailed treatment of supermanifolds to the monograph [8]. To this end, let us consider a local coordinate system $(x) = (x^1, ..., x^N)$ on the base supermanifold M, in the vicinity of a point P. Let the sets $\{e_i\}$ and $\{e^i\}$ be coordinate bases in the tangent space $T_P M$ and the cotangent space $T^*_P M$, respectively. Under a change of coordinates $(x) \to (\bar{x})$, the basis vectors in $T_P M$ and $T^*_P M$ transform according to

$$\bar{e}^i = e^j \frac{\partial \bar{x}^i}{\partial x^j}\,, \quad \bar{e}_i = e_j \frac{\partial_r x^j}{\partial \bar{x}^i}\,.$$

The transformation matrices obey the following relations:

$$\frac{\partial_r \bar{x}^i}{\partial x^k}\frac{\partial_r x^k}{\partial \bar{x}^j} = \delta^i_j\,, \quad \frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^k} = \delta^i_j\,, \quad \frac{\partial_r x^i}{\partial \bar{x}^k}\frac{\partial_r \bar{x}^k}{\partial x^j} = \delta^i_j\,, \quad \frac{\partial \bar{x}^k}{\partial x^j}\frac{\partial x^i}{\partial \bar{x}^k} = \delta^i_j\,.$$

A tensor field of type (n, m) with rank $n + m$ is given by a set of functions $T^{i_1 \ldots i_n}{}_{j_1 \ldots j_m}(x)$, with Grassmann parity $\epsilon(T^{i_1 \ldots i_n}{}_{j_1 \ldots j_m}) = \epsilon(T) + \epsilon_{i_1} + \cdots + \epsilon_{i_n} + \epsilon_{j_1} + \cdots + \epsilon_{j_m}$, which transform under a change of coordinates, $(x) \to (\bar{x})$, according to

$$\begin{aligned}\bar{T}^{i_1 \ldots i_n}{}_{j_1 \ldots j_m} &= T^{l_1 \ldots l_n}{}_{k_1 \ldots k_m} \frac{\partial_r x^{k_m}}{\partial \bar{x}^{j_m}} \cdots \frac{\partial_r x^{k_1}}{\partial \bar{x}^{j_1}} \frac{\partial \bar{x}^{i_n}}{\partial x^{l_n}} \cdots \frac{\partial \bar{x}^{i_1}}{\partial x^{l_1}} \\ &\times (-1)^{\left(\sum_{s=1}^{m-1}\sum_{p=s+1}^{m}\epsilon_{j_p}(\epsilon_{j_s}+\epsilon_{k_s}) + \sum_{s=1}^{n}\sum_{p=1}^{m}\epsilon_{j_p}(\epsilon_{i_s}+\epsilon_{l_s}) + \sum_{s=1}^{n-1}\sum_{p=s+1}^{n}\epsilon_{i_p}(\epsilon_{i_s}+\epsilon_{l_s})\right)}\,.\end{aligned} \tag{3}$$

In particular, it is easy to see that the unit matrix δ^i_j is a tensor field of type $(1, 1)$.

By analogy with tensor analysis on manifolds, on supermanifolds one introduces an operation $\nabla \equiv \overleftarrow{\nabla}$ of covariant differentiation of tensor fields, by the requirement that this operation should map a tensor field of type (n, m) into a tensor field of type $(n, m+1)$, and that, in case one can introduce local Cartesian coordinates, it should reduce to the usual differentiation. On an arbitrary supermanifold M, a covariant derivative is given by a variety of differentiations with respect to separate coordinates, $\nabla = (\overset{M}{\nabla}_i)$. These differentiations are local operations, acting on a tensor field of type (n, m) by the rule

$$\begin{aligned}T^{i_1 \ldots i_n}{}_{j_1 \ldots j_m} \overset{M}{\nabla}_k &= T^{i_1 \ldots i_n}{}_{j_1 \ldots j_m, k} + \sum_{r=1}^{n} T^{i_1 \ldots l \ldots j_n}{}_{j_1 \ldots j_m} \overset{M}{\Gamma}{}^{i_r}{}_{lk} (-1)^{(\epsilon_{i_r}+\epsilon_l)\left(\epsilon_l + \sum_{p=r+1}^{n}\epsilon_{i_p} + \sum_{p=1}^{m}\epsilon_{j_p}\right)} \\ &- \sum_{s=1}^{m} T^{i_1 \ldots i_n}{}_{j_1 \ldots l \ldots j_m} \overset{M}{\Gamma}{}^{l}{}_{j_s k} (-1)^{(\epsilon_{j_s}+\epsilon_l)\sum_{p=s+1}^{m}\epsilon_{j_p}}\,,\end{aligned} \tag{4}$$

where $\overset{M}{\Gamma}{}^k{}_{ij}(x)$ are generalized Christoffel symbols (connection coefficients), subject to the transformation law

$$\overset{M}{\bar{\Gamma}}{}^k{}_{ij} = (-1)^{\epsilon_j(\epsilon_m+\epsilon_i)} \frac{\partial_r \bar{x}^k}{\partial x^l} \overset{M}{\Gamma}{}^l{}_{mn} \frac{\partial_r x^n}{\partial \bar{x}^j} \frac{\partial_r x^m}{\partial \bar{x}^i} + \frac{\partial_r \bar{x}^k}{\partial x^m} \frac{\partial^2_r x^m}{\partial \bar{x}^i \partial \bar{x}^j}\,.$$

In this paper, we restrict the consideration to *symmetric* connections, i.e., those possessing the property

$$\overset{M}{\Gamma}{}^{k}{}_{ij} = (-1)^{\epsilon_i \epsilon_j} \overset{M}{\Gamma}{}^{k}{}_{ji}\,.$$

Note that this property is fulfilled automatically in case a local Cartesian system can be introduced on the supermanifold M.

The curvature tensor $\overset{M}{R}{}^{i}{}_{mjk}(x)$ is defined by the action of the (generalized) commutator of covariant derivatives $[\overset{M}{\nabla}_i, \overset{M}{\nabla}_j] = \overset{M}{\nabla}_i \overset{M}{\nabla}_j - (-1)^{\epsilon_i \epsilon_j} \overset{M}{\nabla}_j \overset{M}{\nabla}_i$ on a vector field T^i by the rule

$$T^i [\overset{M}{\nabla}_j, \overset{M}{\nabla}_k] = -(-1)^{\epsilon_m(\epsilon_i + 1)} T^m \overset{M}{R}{}^{i}{}_{mjk}\,.$$

A straightforward calculation yields the following result:

$$\overset{M}{R}{}^{i}{}_{mjk} = -\overset{M}{\Gamma}{}^{i}{}_{mj,k} + \overset{M}{\Gamma}{}^{i}{}_{mk,j}(-1)^{\epsilon_j \epsilon_k} + \overset{M}{\Gamma}{}^{i}{}_{jl} \overset{M}{\Gamma}{}^{l}{}_{mk}(-1)^{\epsilon_j \epsilon_m} - \overset{M}{\Gamma}{}^{i}{}_{kl} \overset{M}{\Gamma}{}^{l}{}_{mj}(-1)^{\epsilon_k(\epsilon_m + \epsilon_j)}. \tag{5}$$

The curvature tensor (5) possesses the property of generalized symmetry

$$\overset{M}{R}{}^{i}{}_{mjk} = -(-1)^{\epsilon_j \epsilon_k} \overset{M}{R}{}^{i}{}_{mkj}$$

and obeys the Jacobi identity

$$(-1)^{\epsilon_j \epsilon_l} \overset{M}{R}{}^{i}{}_{jkl} + \mathrm{cycle}\,(j,k,l) \equiv 0.$$

On triplectic supermanifolds $\mathcal{M}$, one defines [6] covariant differentiation of tensor fields transforming as tensors on the base supermanifold M. In a similar way, we introduce a superfield extension of the triplectic covariant derivative. Having in mind the coordinate transformations (1) on the supermanifold $\mathcal{N}$, we define a tensor field of type (n,m) and rank $n+m$ as a geometric object which in any local coordinate system (x, θ, y) is given by a set of functions $T^{i_1 \ldots i_n}{}_{j_1 \ldots j_m}(z)$ transforming by the tensor law (3). Let us define the superfield covariant derivative $\mathcal{D} \equiv \overleftarrow{\mathcal{D}}$ in a Cartesian coordinate system to coincide with $\frac{\overleftarrow{\partial}}{\partial z^i(\eta)}$, given by (2). Then, in general coordinates, $\mathcal{D} = (\mathcal{D}_i(\eta))$ becomes

$$\overleftarrow{\mathcal{D}}_i(\eta) = \frac{\overleftarrow{\partial}}{\partial \theta^i_a}\,\eta_a + \overset{\mathcal{M}}{\overleftarrow{\nabla}}_i\,\eta^2.$$

Here, each term of the η-expansion acts as a covariant differentiation of tensor fields $T^{i_1 \ldots i_n}{}_{j_1 \ldots j_m}(z)$. The component $\overset{\mathcal{M}}{\nabla}_i$ is an extension of the covariant derivative $\overset{M}{\nabla}_i$, given by (4) on the base supermanifold M, namely,

$$\overset{\mathcal{M}}{\overleftarrow{\nabla}}_i = \overset{M}{\overleftarrow{\nabla}}_i - \frac{\overleftarrow{\partial}}{\partial \theta^k_a}\theta^m_a \overset{M}{\Gamma}{}^{k}{}_{mi}(-1)^{\epsilon_m(\epsilon_k + 1)}. \tag{6}$$

The operation $\overset{\mathcal{M}}{\nabla}_i$ coincides[1] with the triplectic covariant derivative [6].

Since by definition (x^i, θ^i_a) are independent coordinates, (4), (6) imply that the vectors θ^i_a are covariantly constant with respect to $\overset{\mathcal{M}}{\nabla}_i$, namely,

$$\theta^i_a \overset{\mathcal{M}}{\nabla}_j = 0. \tag{7}$$

By virtue of (4), (6), the commutator of two superfield covariant derivatives $\mathcal{D}_i(\eta)$ has the form

$$\left[\mathcal{D}_i(\eta), \mathcal{D}_j(\eta')\right] = [\overset{\mathcal{M}}{\nabla}_i, \overset{\mathcal{M}}{\nabla}_j]\,\eta^2\,(\eta')^2\,.$$

[1]To observe the coincidence of (6) with the triplectic covariant derivative [6], one should go over to the parameterization (x^i, θ_{ia}), where θ_{ia} transform as vectors of the tangent space $T_P M$ (for details, see Section 3.4).

¿From (6), (7), it follows that the action of this commutator on a scalar field $T = T(z)$ is given by

$$T\left[\mathcal{D}_i(\eta), \mathcal{D}_j(\eta')\right] = (-1)^{\epsilon_m(\epsilon_n+1)}\eta^2\left(\eta'\right)^2 \frac{\partial_r T}{\partial\theta^n_a}\theta^m_a \overset{M}{R}{}^n{}_{mij}\,,$$

where $\overset{M}{R}{}^n{}_{mij}$ is the curvature tensor (5) on the base supermanifold.

3 Superfield Realization of (Modified) Triplectic Algebra

The extended BRST quantization in general coordinates [6] is based on a realization of the so-called triplectic [3] and modified triplectic [4] operator algebras. The operators obeying these algebras are originally defined on triplectic supermanifolds $\mathcal{M}$. In this section, we propose a superfield formulation of [6], realized on extended supermanifolds $\mathcal{N}$. Namely, we construct a manifestly superfield realization of the (modified) triplectic algebra, which permits us to formulate a superfield realization of extended BRST quantization in general coordinates, along the lines of [6].

3.1 Triplectic and Modified Triplectic Algebras

The triplectic algebra [3] includes two sets of second- and first-order operators, $\overleftarrow{\Delta}^a$ and $\overleftarrow{V}^a$, respectively, having the Grassmann parity $\epsilon(\Delta^a) = \epsilon(V^a) = 1$, and obeying the following relations:

$$\Delta^{\{a}\Delta^{b\}} = 0, \quad V^{\{a}V^{b\}} = 0, \quad V^a\Delta^b + \Delta^b V^a = 0. \tag{8}$$

The modified triplectic quantization [4], in comparison with the $Sp(2)$-covariant approach [1] and the triplectic scheme [3], involves an additional $Sp(2)$-doublet of first-order operators $\overleftarrow{U}^a$, $\epsilon(U^a) = 1$, with the modified triplectic algebra [4] given by the relations

$$\begin{gathered}\Delta^{\{a}\Delta^{b\}} = 0, \quad V^{\{a}V^{b\}} = 0, \quad U^{\{a}U^{b\}} = 0,\\ V^{\{a}\Delta^{b\}} + \Delta^{\{b}V^{a\}} = 0, \quad \Delta^{\{a}U^{b\}} + U^{\{a}\Delta^{b\}} = 0, \quad U^{\{a}V^{b\}} + V^{\{a}U^{b\}} = 0.\end{gathered} \tag{9}$$

In (8), (9), the curly brackets denote symmetrization with respect to the enclosed indices a and b.

Using the odd second-order differential operators Δ^a, one can introduce a pair of bilinear operations $(\ ,\ )^a$, by the rule

$$(F, G)^a = (-1)^{\epsilon(G)}(FG)\Delta^a - (-1)^{\epsilon(G)}(F\Delta^a)G - F(G\Delta^a). \tag{10}$$

The operations (10) possess the Grassmann parity $\epsilon((F,G)^a) = \epsilon(F) + \epsilon(G) + 1$ and obey the following symmetry property:

$$(F, G)^a = -(-1)^{(\epsilon(G)+1)(\epsilon(F)+1)}(G, F)^a.$$

The operations (10) are linear with respect to both arguments,

$$(F + G, H)^a = (F, H)^a + (G, H)^a, \quad (F, G + H)^a = (F, G)^a + (F, H)^a,$$

and obey the Leibniz rule

$$(F, GH)^a = (F, G)^a H + (F, H)^a G(-1)^{\epsilon(G)\epsilon(H)}.$$

Due to the properties (8) of the operators Δ^a, the odd bracket operations satisfy the generalized Jacobi identity

$$(F, (G, H)^{\{a})^{b\}}(-1)^{(\epsilon(F)+1)(\epsilon(H)+1)} + \mathrm{cycle}(F, G, H) \equiv 0.$$

In view of their properties, the operations $(\ ,\ )^a$ form a set of antibrackets, such as those introduced for the first time in [1]. Therefore, having an explicit realization of operators Δ^a with the properties (8), one can generate the extended antibrackets explicitly, using (10). Explicit realizations of Δ^a are known in two cases: in Darboux coordinates [1, 3, 4], and in general coordinates on triplectic supermanifolds $\mathcal{M}$, where the base supermanifold M is a flat Fedosov supermanifold [6, 9].

3.2 Realization of Triplectic Algebra

To find an explicit superfield realization of the triplectic algebra (8) in general coordinates, we shall use the assumptions of [6] concerning the properties of the base supermanifold M. Thus, we equip M with a Poisson structure, namely, with a nondegenerate *even* second-rank tensor field $\omega^{ij}(x)$, and its inverse $\omega_{ij}(x)$, $\epsilon(\omega^{ij}) = \epsilon(\omega_{ij}) = \epsilon_i + \epsilon_j$,

$$\omega^{ik}\omega_{kj}(-1)^{\epsilon_k} = \delta^i_j\,, \quad \omega_{ik}\omega^{kj}(-1)^{\epsilon_i} = \delta^j_i\,,$$

obeying the properties of generalized antisymmetry

$$\omega^{ij} = -(-1)^{\epsilon_i\epsilon_j}\omega^{ji} \Leftrightarrow \omega_{ij} = -(-1)^{\epsilon_i\epsilon_j}\omega_{ji},$$

and satisfying the following Jacobi identities:

$$\omega^{il}\partial_l\omega^{jk}(-1)^{\epsilon_i\epsilon_k} + \mathrm{cycle}(i,j,k) \equiv 0 \Leftrightarrow \omega_{ij,k}(-1)^{\epsilon_i\epsilon_k} + \mathrm{cycle}(i,j,k) \equiv 0.$$

The tensor field ω^{ij} defines a Poisson bracket [6], and, due to its nondegeneracy, also a corresponding *even* symplectic structure [6] on the base supermanifold. In view of this fact, the supermanifold M can be regarded as an even Poisson supermanifold, as well as an even symplectic supermanifold. Following [6], we demand that the covariant derivative $\overset{M}{\nabla}_i$ should respect the Poisson structure ω^{ij},

$$\omega^{ij}\overset{M}{\nabla}_k = 0 \Leftrightarrow \omega_{ij}\overset{M}{\nabla}_k = 0, \tag{11}$$

which provides the covariant constancy of the differential two-form $\omega = \omega_{ij}dx^j \wedge dx^i$. Thus, the base supermanifold M can be regarded as an even symplectic supermanifold, being a supersymmetric extension [6, 9] of the Fedosov manifold [10, 11]. One can formally identify ω^{ij} and ω_{ij} with some functions of the supervariables $z^i(\eta)$, i.e., $\Omega^{ij}(z) = \omega^{ij}(x)$ and $\Omega_{ij}(z) = \omega_{ij}(x)$. It is obvious that the tensor fields Ω^{ij} and Ω_{ij} are covariantly constant:

$$\Omega^{ij}\mathcal{D}_k(\eta) = \Omega_{ij}\mathcal{D}_k(\eta) = 0.$$

The introduced structures allow one to equip the supermanifold $\mathcal{N}$ with a superfield $Sp(2)$-irreducible second-rank tensor S_{ab},

$$S_{ab} = \frac{1}{6}\int d^2\eta\, \eta^2 \frac{\partial z^i}{\partial\eta^a}\Omega_{ij}\frac{\partial z^j}{\partial\eta^b}\,, \quad \epsilon(S_{ab}) = 0, \tag{12}$$

invariant under changes of local coordinates on $\mathcal{N}$, i.e., $\bar{S}_{ab} = S_{ab}$, and symmetric with respect to the $Sp(2)$-indices, $S_{ab} = S_{ba}$.

Following [5, 6], we also equip the base supermanifold M with a scalar density $\rho(x)$, $\epsilon(\rho) = 0$. Using the covariant derivative $\mathcal{D}_i(\eta)$, we can construct a superfield $Sp(2)$-doublet of odd second-order differential operators Δ^a, acting as scalars on the supermanifold $\mathcal{N}$,

$$\overleftarrow{\Delta}^a = \int d^2\eta\, \eta^2 \left(\overleftarrow{\mathcal{D}}_i \frac{\partial_r}{\partial\eta^a}\right)\Omega^{ij}\left[\left(\overleftarrow{\mathcal{D}}_j + \frac{1}{2}(\mathcal{R}\overleftarrow{\mathcal{D}}_j)\right)\frac{\partial_r}{\partial\eta^2}\right](-1)^{\epsilon_i+\epsilon_j}, \tag{13}$$

where $\mathcal{R}(z) \equiv \rho(x)$.

The operators (13) generate a superfield $Sp(2)$-doublet of antibracket operations,

$$(F,G)^a = -\int d^2\eta\, \eta^2 \left(F\mathcal{D}_i\frac{\partial_r}{\partial\eta^2}\right)\Omega^{ij}\frac{\partial}{\partial\eta^a}(G\mathcal{D}_j)(-1)^{\epsilon_j\epsilon(G)} - (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}(F \leftrightarrow G). \tag{14}$$

These operations possess all the properties of extended antibrackets [1], except the Jacobi identity, which is closely related to the properties (8) of anticommutativity and nilpotency of Δ^a.

Using the operations (14) and the irreducible second-rank $Sp(2)$-tensor S_{ab} in (12), we define the following $Sp(2)$-doublet of odd first-order differential operators V_a:

$$\overleftarrow{V}_a = (\,\cdot\,, S_{ab})^b = -\frac{1}{2}\int d^2\eta\, \eta^2 \left(\overleftarrow{\mathcal{D}}_i\frac{\partial_r}{\partial\eta^2}\right)\frac{\partial_r z^i}{\partial\eta^a}\,. \tag{15}$$

Straightforward calculations, analogous to [6], with allowance for the manifest form of the operators Δ^a, V^a, (13), (15), show that there exists such a choice of the density function $\mathcal{R}$,

$$\mathcal{R} = -\log \text{sdet} \left(\Omega^{ij}\right),$$

that the triplectic algebra (8) is fulfilled on $\mathcal{N}$ in case the base supermanifold M is a flat Fedosov supermanifold:

$$\overset{M}{R}{}^{i}{}_{mjk} = 0,$$

with the curvature tensor $\overset{M}{R}{}^{i}{}_{mjk}$ given by (5). Thus, we have explicitly realized the extended antibrackets (14) and the triplectic algebra (8) of the generating operators Δ^a, V^a.

3.3 Realization of Modified Triplectic Algebra

In view of (8), to complete the explicit superfield realization of the modified triplectic algebra (9) in general coordinates, it remains to construct the operators U^a. To this end, following [6], we introduce another geometrical structure on the base supermanifold M. Namely, we consider a symmetric second-rank tensor $g_{ij}(x) = (-1)^{\epsilon_i\epsilon_j} g_{ji}(x)$, which we identify with a tensor field $G_{ij}(z)$. The introduced tensor field can be used to construct on $\mathcal{N}$ an $Sp(2)$ scalar function S_0, the so-called anti-Hamiltonian,

$$S_0 = \frac{1}{2}\varepsilon^{ab}\int d^2\eta\,\eta^2 \frac{\partial_r z^i}{\partial\eta^a}\,G_{ij}\,\frac{\partial_r z^j}{\partial\eta^b}\,,\quad \epsilon(S_0) = 0. \tag{16}$$

The anti-Hamiltonian S_0 generates vector fields U^a,

$$\overleftarrow{U}{}^a = (\,\cdot, S_0)^a = \int d^2\eta\,\eta^2\left[\left(\overleftarrow{\mathcal{D}}_i\frac{\partial_r}{\partial\eta^2}\right)\Omega^{im}G_{mn}\frac{\partial z^n}{\partial\eta_a}\,(-1)^{\epsilon_m} + \frac{1}{2}\left(\overleftarrow{\mathcal{D}}_i\frac{\partial_r}{\partial\eta^a}\right)\Omega^{ij}\frac{\partial_r z^m}{\partial\eta^c}\left(G_{mn}\overleftarrow{\mathcal{D}}_j\frac{\partial_r}{\partial\eta^2}\right)\frac{\partial_r z^n}{\partial\eta_c}(-1)^{\epsilon_i+\epsilon_j\epsilon_n}\right].$$

The algebraic conditions (9) yield the following equations for S_0:

$$(S_0, S_0)^a = 0,\quad S_0V^a = 0,\quad S_0\Delta^a = 0. \tag{17}$$

Solutions of these equations always exist. An example of such soultions can be found in the class of covariantly constant[2] tensor fields G_{ij}, $G_{ij}\mathcal{D}_k = 0$. We do not restrict ourselves to this special case, and simply assume that equations (17) are fulfilled. Thus, we obtain a realization of the modified triplectic algebra (9), and have at our disposal all the ingredients for the quantization of general gauge theories within the modified triplectic scheme.

3.4 Quantization

The quantization procedure repeats all the essential steps taken for the first time in [5], and leads to the vacuum functional

$$Z = \int dz\,\mathcal{D}_0\exp\{(i/\hbar)[W + X + \alpha S_0]\}, \tag{18}$$

where α is an arbitrary constant; the function S_0 is given by (16), while the quantum action $W = W(z)$ and the gauge-fixing functional $X = X(z)$ satisfy the following quantum master equations:

$$\frac{1}{2}(W,W)^a + W\mathcal{V}^a = i\hbar W\Delta^a, \tag{19}$$

$$\frac{1}{2}(X,X)^a + X\mathcal{U}^a = i\hbar X\Delta^a. \tag{20}$$

[2]In the class of covariantly constant tensors G_{ij}, solutions of (17) can be selected by imposing the condition $G_{ij}\,(\mathcal{R}\mathcal{D}_k)\,\Omega^{kj} = 0$. The simplest solution of this kind is given by a covariantly constant scalar density $\mathcal{R}$.

In (18), integration over the supervariables is understood as integration over their components,

$$dz = dx\, d\theta_a\, dy,$$

with the integration measure $\mathcal{D}_0$ given by

$$\mathcal{D}_0 = \left[\mathrm{sdet}\left(\Omega^{ij}\right)\right]^{-3/2}.$$

In (19) and (20), we have introduced operators $\mathcal{V}^a$, $\mathcal{U}^a$, according to

$$\mathcal{V}^a = \frac{1}{2}\left(\alpha U^a + \beta V^a + \gamma U^a\right), \quad \mathcal{U}^a = \frac{1}{2}\left(\alpha U^a - \beta V^a - \gamma U^a\right).$$

It is obvious that for arbitrary constants α, β, γ the operators $\mathcal{V}^a$, $\mathcal{U}^a$ obey the properties

$$\mathcal{V}^{\{a}\mathcal{V}^{b\}} = 0, \quad \mathcal{U}^{\{a}\mathcal{U}^{b\}} = 0, \quad \mathcal{V}^{\{a}\mathcal{U}^{b\}} + \mathcal{U}^{\{a}\mathcal{V}^{b\}} = 0.$$

Therefore, the operators Δ^a, $\mathcal{V}^a$, $\mathcal{U}^a$ also realize the modified triplectic algebra.

The integrand of the vacuum functional (18) is invariant under extended BRST transformations defined by the generators

$$\delta^a = (\,\cdot, W - X)^a + \mathcal{V}^a - \mathcal{U}^a. \tag{21}$$

In the usual manner, this allows one to prove that, for every given set of the parameters α, β, γ, the vacuum functional (18) does not depend on a choice of the gauge-fixing function X.

Let us analyze the component structure of the proposed quantization scheme in order to establish its relation with the modified triplectic quantization in general coordinates [6]. To this end, note that the integration measure $\mathcal{D}_0$ and the function S_0,

$$S_0 = \frac{1}{2}\varepsilon^{ab}\theta^i_a\, g_{ij}\theta^j_b(-1)^{\epsilon_i+\epsilon_j},$$

coincide with the corresponding objects of [6]. The operators Δ^a, V^a, U^a and antibrackets $(\ ,\ )^a$ have the form

$$\begin{aligned}
\overleftarrow{\Delta}^a &= (-1)^{\epsilon_i}\frac{\overleftarrow{\partial}}{\partial\theta_{ia}}\left(\overset{\mathcal{M}}{\overleftarrow{\nabla}}_i + \frac{1}{2}\rho_{,i}\right),\\
\overleftarrow{V}^a &= \frac{1}{2}\epsilon^{ab}\overset{\mathcal{M}}{\overleftarrow{\nabla}}_i\omega^{ij}\theta_{jb},\\
\overleftarrow{U}^a &= -\overset{\mathcal{M}}{\overleftarrow{\nabla}}_i\omega^{im}g_{mn}\theta^{na}(-1)^{\epsilon_m} - \frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\theta_{ia}}\theta^m_c(g_{mn}\overset{\mathcal{M}}{\overleftarrow{\nabla}}_i)\theta^{nc}(-1)^{\epsilon_n(\epsilon_i+1)+\epsilon_m},\\
(F,G)^a &= (F\overset{\mathcal{M}}{\overleftarrow{\nabla}}_i)\frac{\partial G}{\partial\theta_{ia}} - (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}(G\overset{\mathcal{M}}{\overleftarrow{\nabla}}_i)\frac{\partial F}{\partial\theta_{ia}},
\end{aligned}$$

where θ_{ia}, defined by $\theta^i_a = \omega^{ij}\theta_{ja}(-1)^{\epsilon_i}$, are covariantly constant covectors, $\theta_{ia}\overset{\mathcal{M}}{\overleftarrow{\nabla}}_j = 0$, while $\frac{\overleftarrow{\partial}}{\partial\theta_{ia}}$ transform as vectors. The above component expressions imply that the operators Δ^a, V^a, U^a and antibrackets coincide with the corresponding objects of [6], which follows from the coincidence of $\overset{\mathcal{M}}{\overleftarrow{\nabla}}_i$ with the triplectic covariant derivative [6], given by

$$\overset{\mathcal{M}}{\overleftarrow{\nabla}}_i = \overset{M}{\overleftarrow{\nabla}}_i + \frac{\overleftarrow{\partial}}{\partial\theta_{ma}}\theta_{ka}\overset{M}{\Gamma}{}^k{}_{mi}.$$

In case local Cartesian coordinates can be introduced on M, the coincidence of derivatives is automatic, while in the case of arbitrary connection coefficients, the coincidence takes place since M is a Fedosov supermanifold, namely, due to (11). Equations (19), (20) formally coincide with the master equations of [6], because the external variables y^i enter only as arguments of $W(z)$ and $X(z)$. In Darboux coordinates $(\tilde{z}^\mu, y^i)$, $y^i = (\lambda^A, J_A)$, one can choose solutions of (19), (20) as solutions of the master equations [6], namely, $W = W(\tilde{z})$, $X = X(\tilde{z},\lambda)$. Since in the coordinates $(\tilde{z}^\mu, y^i)$ the tensor ω^{ij} can be chosen [5] such that $\mathcal{D}_0 = \mathrm{const}$, the vacuum functional (18) reduces to

$$Z = \int d\tilde{z}\, d\lambda \exp\{(i/\hbar)[W(\tilde{z}) + X(\tilde{z},\lambda) + \alpha S_0(\tilde{z})]\},$$

which is identical with the vacuum functional [6], written in Darboux coordinates.

4 Conclusion

In this paper, we have proposed a superfield realization of extended BRST symmetry in general coordinates, along the lines of our recent works [5, 6] on modified triplectic quantization. We have found an explicit superfield realization of the modified triplectic algebra of generating operators Δ^a, V^a, U^a on an extended supermanifold $\mathcal{N}$, obtained from the triplectic supermanifold $\mathcal{M}$ by adding external supervariables, which, in Darboux coordinates, can be interpreted as sources J_A to the fields and as auxiliary gauge-fixing variables λ^A. The present study applies the essential ingredients of [5, 6], and has the same general features. Thus, the base supermanifold M of fields and antifields is a flat Fedosov supermanifold equipped with a symmetric structure. As in [5, 6], the formalism is characterized by free parameters, $(\alpha,\ \beta,\ \gamma)$, whose specific choice in Darboux coordinates reproduces all the known schemes of covariant quantization based on extended BRST symmetry (for details, see [5]). Every specific choice of the free parameters $(\alpha,\ \beta,\ \gamma)$ yields a gauge-independent vacuum functional and, therefore, a gauge independent S-matrix (see [12]).

Acknowledgments: The authors are grateful to D.V. Vassilevich for stimulating discussions. The work was supported by Deutsche Forschungsgemeinschaft (DFG), grant GE 696/7-1. D.M.G. acknowledges the support of the foundations FAPESP, CNPq and DAAD. The work of P.M.L. was supported by the Russian Foundation for Basic Research (RFBR), 02-02-04002, 03-02-16193, and by the President grant 1252.2003.2 for Support of Leading Scientific Schools. P.Yu.M. is grateful to FAPESP, grant 02/00423-4.

References

[1] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, J. Math. Phys. **31**, 1487 (1990); **32**, 532 (1990); **32**, 2513 (1990);
P.M. Lavrov, Mod. Phys. Lett. **A6**, 2051 (1991); *Theor. Math. Phys.* **89**, 1187 (1991).

[2] P.M. Lavrov, *Phys. Lett.* **B366**, 160 (1996); *Theor. Math. Phys.* **107**, 602 (1996);
P.M. Lavrov and P.Yu. Moshin, *Phys. Lett.* **B508**, 127 (2001); *Theor. Math. Phys.* **129**, 1645 (2001).

[3] I.A. Batalin and R. Marnelius, *Phys. Lett.* **B350**, 44 (1995); *Nucl. Phys.* **B465**, 521 (1996);
I.A. Batalin, R. Marnelius and A.M. Semikhatov, *Nucl. Phys.* **B446**, 249 (1995).

[4] B. Geyer, D.M. Gitman and P.M. Lavrov, *Mod. Phys. Lett.* **A14**, 661 (1999); *Theor. Math. Phys.* **123**, 813 (2000).

[5] B. Geyer, P. Lavrov and A. Nersessian, *Phys. Lett.* **B512**, 211 (2001); *Int. J. Mod. Phys.* **A17**, 1183 (2002).

[6] B. Geyer and P. Lavrov, *Modified triplectic quantization in general coordinates*, hep-th/0304011.

[7] B.S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).

[8] B. DeWitt, *Supermanifolds,* 2nd ed. (Cambridge University Press, Cambridge, 1992).

[9] B. Geyer and P. Lavrov, *Fedosov supermanifolds: basic properties and the difference in even and odd cases*, hep-th/0306218.

[10] B.V. Fedosov, *J. Diff. Geom.* **40**, 213 (1994);
B.V. Fedosov, *Deformation Quantization and Index Theory* (Akademie Verlag, Berlin, 1996).

[11] I. Gelfand, V. Retakh and M. Shubin, *Advan. Math.* **136**, 104 (1998); dg-ga/9707024.

[12] I.V. Tyutin, *Phys. Atom. Nucl.* **65**, 194 (2002).